Pergamon

Int. J. Solids Structures Vol. 35, No. 25, pp. 3317–3330, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00181–9

METHODS FOR CALCULATING STRESS INTENSITY FACTORS FOR INTERFACIAL CRACKS BETWEEN TWO ORTHOTROPIC SOLIDS

W. QIAN and C. T. SUN*
School of Aeronautics and Astronautics, Purdue University, West Lafayette, IN 47907, U.S.A.

(*Received* 12 *November* 1996; *in revised form* 17 *June* 1997)

Abstract—Stress intensity factors for interfacial cracks between two dissimilar orthotropic materials were considered. Due to the oscillatory characteristics of stresses and displacements near the crack tip, individual strain energy release rates no longer exist. Instead, the individual strain energy release rates corresponding to a finite crack extension were obtained in terms of the stress intensity factors and the assumed crack extension Δa. The finite element methods in conjunction with the crack closure technique were used to calculate these finite extension strain energy release rates from which accurate stress intensity factors were obtained. An alternative method based on crack surface displacement ratio was also discussed. Non-oscillatory (Δa-independent) Mode I and Mode II "strain energy release rates" were also proposed to provide an alternate measure of fracture mode mixity or to be used as a fracture criterion for interfacial cracks. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Interfacial cracking is a common failure mode in bimaterial media. Examples include debonding of adhesive joints, delamination in advanced composite materials and their laminates, and grain boundary fractures in polycrystals resulting from precipitation or segregation of impurities. To predict the growth of an interfacial crack, the neartip stress and displacement fields must be calculated.

Since Williams (1959) discovered the oscillatory neartip stress behavior for a traction free interfacial crack between two dissimilar isotropic materials, the interface crack problem has been discussed by many authors such as England (1965), Rice and Sih (1965), Malysev and Salganik (1965), Sun and Jih (1987), Hutchinson *et al.* (1987), Rice (1988) for isotropic media; and Gotoh (1967), Clements (1971), Willis (1971), Wang and Choi (1983a, 1983b), Ting (1986), Bassani and Qu (1989), Sun and Manoharan (1989), Wu (1990), Gao *et al.* (1992), and Hwu (1993a, 1993b) for anisotropic media. The oscillatory stress and displacement fields are physically inadmissible due to the wrinkle and over-lap zone near the end of the crack (England, 1965). Under remote tensile loading, this zone is generally very small compared to the crack length. However, it can be large for shear loading (see Rice, 1988; Comninou, 1978). Modifications to account for crack surface contact have been suggested to resolve this dilemma by Comninou (1977). Rice (1988) argued for the validity of oscillatory field solutions by proposing the so called small scale contact zone outside which the oscillatory solution adequately describes the near-tip state. This was also confirmed recently by Sun and Qian (1996a) through numerical comparison of the near-tip solution of the oscillatory model and the contact model for interfacial cracks under remote uniform tensile loading in isotropic media. The limitation of the loading condition in isotropic media was also studied by Sun and Qian (1997) and given in terms of the phase of the complex stress intensity factor.

Although the total strain energy release rate for interfacial cracks is well defined and has been found explicitly in both isotropic and anisotropic media, the individual strain energy release rates G_{I} and G_{II} do not exist due to their oscillatory nature (see Sun and Jih, 1987; Raju *et al.*, 1988). However, for a finite extension Δa, the individual strain energy release rates exist and were used by Sun and Qian (1997) to calculate stress intensity factors

* Author to whom correspondence should be addressed.

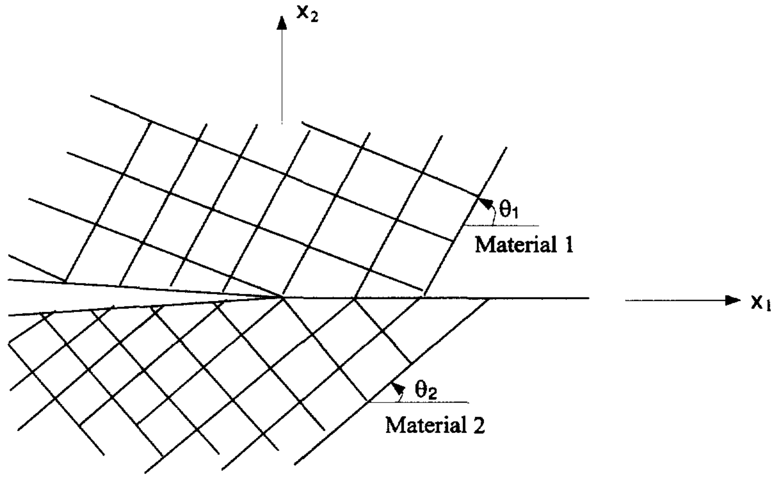

Fig. 1. Interfacial crack between two dissimilar orthotropic materials having one plane of material symmetry parallel to the x_1–x_2 plane.

for interfacial cracks in isotropic media and by Qian and Sun (1995, 1996b) for interfacial cracks in composite laminates.

In the present study, the aforementioned approach is extended to general orthotropic bimaterial media having one plane of material symmetry parallel to x_1–x_2, plane, see Fig. 1. Owing to the transformation property of the Barnett–Lothe tensors (Barnett and Lothe, 1973) derived by Dongye and Ting (1989) for these particular bimaterial media, the explicitly near-tip stresses and displacements are obtained from the solution by Hwu (1993b). The crack closure integrals are evaluated for a finite crack extension, from which the stress intensity factors are derived. The ratio of stress intensity factors can also be evaluated alternatively from the crack surface displacements ratio. These stress intensity ratios are used in conjunction with the total strain energy release rate to determine the stress intensity factors. Non-oscillatory (Δa-independent) strain energy release rates are also proposed and their relations to finite extension strain energy release rates are presented. The mode mixity is expressed in terms of either the phase of the complex stress intensity factor or the ratio of the two non-oscillatory strain energy release rates. A center crack lying between two dissimilar orthorhombic crystalline materials with different material principal orientations under remote uniform tensile loading is selected in the numerical study. Numerical results are presented to show the accuracy of this technique.

2. NEAR-TIP FIELD

Consider an interfacial crack located between two orthotropic media. Each medium is considered as an orthotropic medium rotated with a θ in the x_1–x_2 plane, see Fig. 1. The x_3-axis coincides with a material principal axis, and loading is independent of the x_3-axis, resulting in a generalized plane strain deformation. The stress intensity factors associated with an arbitrary distance $\hat{r}$ introduced by Wu (1990) are expressed by Hwu (1993a) as

$$\begin{Bmatrix} K_{\mathrm{II}} \\ K_{\mathrm{I}} \\ K_{\mathrm{III}} \end{Bmatrix} = \lim_{r\to 0} \sqrt{2\pi r}\,\mathbf{\Lambda}\langle\langle (r/\hat{r})^{-i\varepsilon_\alpha}\rangle\rangle \mathbf{\Lambda}^{-1} \begin{Bmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{23} \end{Bmatrix} \tag{1}$$

where the angular brackets $\langle\langle\ \rangle\rangle$ stand for a 3×3 diagonal matrix, ε_α ($\alpha = 1, 2, 3$) involve the bimaterial constant, and $\mathbf{\Lambda}$ is the eigenvector matrix associated with the eigenvalue problem as discussed in Hwu (1993a) (see Appendix). The stress intensity factors obtained at one $\hat{r}$ can be converted to those at a different $\hat{r}$. To be consistent with the definition of stress intensity factor by Sun and Jih (1987) in isotropic media, we chose crack length $2a$

as the value of $\hat{r}$ throughout the paper, and stress intensity factor at other $\hat{r}$ can easily be obtained through the following relation

$$K(\hat{r}) = \mathbf{\Lambda}\langle\langle(2a/\hat{r})^{-i\varepsilon_\alpha}\rangle\rangle\mathbf{\Lambda}^{-1}K(2a) \tag{2}$$

The near-tip stresses ahead of the crack tip and relative crack surface displacements have been derived by Hwu (1993) as

$$\begin{Bmatrix}\sigma_{12}\\ \sigma_{22}\\ \sigma_{23}\end{Bmatrix} = \frac{1}{\sqrt{2\pi r}}\mathbf{\Lambda}\langle\langle(r/2a)^{i\varepsilon_\alpha}\rangle\rangle\mathbf{\Lambda}^{-1}\begin{Bmatrix}K_{\mathrm{II}}\\ K_{\mathrm{I}}\\ K_{\mathrm{III}}\end{Bmatrix} \tag{3}$$

and

$$\begin{Bmatrix}\Delta u_1\\ \Delta u_2\\ \Delta u_3\end{Bmatrix} = \sqrt{\frac{2r}{\pi}}(\bar{\mathbf{\Lambda}}^T)^{-1}\left\langle\left\langle\frac{(r/2a)^{i\varepsilon_\alpha}}{(1+2i\varepsilon_\alpha)\cosh(\pi\varepsilon_\alpha)}\right\rangle\right\rangle\mathbf{\Lambda}^{-1}\begin{Bmatrix}K_{\mathrm{II}}\\ K_{\mathrm{I}}\\ K_{\mathrm{III}}\end{Bmatrix} \tag{4}$$

respectively. In (4), the overbar stands for the conjugate of a complex number. It is seen that the influence of material properties on near-tip stresses and displacement fields for the interfacial crack are reflected through the oscillation index ε_α ($\alpha = 1, 2, 3$) and eigenvector matrix $\mathbf{\Lambda}$, which are obtained in the standard eigenvalue problem given in (A1) with two known 3×3 real matrices $\mathbf{D}$ and $\mathbf{W}$ or Barnett and Lothe tensors $\mathbf{S}$ and $\mathbf{L}$ (Barnett and Lothe, 1973) (see Appendix). The explicit expression for $\mathbf{S}$ and $\mathbf{L}$ of orthotropic materials have been shown by Dongye and Ting (1989) as

$$S_{21} = \left[\frac{C_{66}(\sqrt{C_{11}C_{22}}-C_{12})}{C_{22}(C_{12}+2C_{66}+\sqrt{C_{11}C_{22}})}\right]^{1/2}, \quad S_{12} = -\sqrt{\frac{C_{22}}{C_{11}}}S_{21}$$

$$L_{11} = (C_{12}+\sqrt{C_{11}C_{22}})S_{21}, \quad L_{22} = \sqrt{\frac{C_{22}}{C_{11}}}L_{11}, \quad L_{33} = (C_{44}C_{55})^{1/2} \tag{5}$$

All other elements of $\mathbf{S}$ and $\mathbf{L}$ are zero. In the above , C_{ij} is the contracted notation for the fourth order elastic constant tensor C_{ijks}.

For an interfacial crack between two dissimilar orthotropic materials having one plane of material symmetry parallel to the x_1–x_2 plane as illustrated in Fig. 1, Dongye and Ting (1989) showed the following transformation relations between the transformed $\mathbf{S}(\boldsymbol{\theta})$, $\mathbf{L}(\boldsymbol{\theta})$ and $\mathbf{S}$, $\mathbf{L}$ as

$$\mathbf{S}(\theta) = \mathbf{\Omega}(\theta)\mathbf{S}\mathbf{\Omega}^T(\theta)$$
$$\mathbf{L}(\theta) = \mathbf{\Omega}(\theta)\mathbf{L}\mathbf{\Omega}^T(\theta) \tag{6}$$

and the coordinate transformation matrix $\mathbf{\Omega}(\Delta a)$ is given by

$$\mathbf{\Omega}(\theta) = \begin{bmatrix}\cos(\theta) & \sin(\theta) & 0\\ -\sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}$$

and θ is the angle of rotation of the orthotropy axes in the x_1–x_2 plane.

The $\mathbf{D}$ and $\mathbf{W}$ matrices can be obtained explicitly by substituting (6) into (A2). We obtain

$$\mathbf{D} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} \tag{7}$$

$$\mathbf{W} = \mathbf{S}_1\mathbf{L}_1^{-1} - \mathbf{S}_2\mathbf{L}_2^{-1} = \begin{bmatrix} 0 & -W_{21} & 0 \\ W_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

where the subscripts 1 and 2 on **S** and **L** denote the upper and lower media, respectively, and

$$\begin{aligned} D_{11} &= \sum_{i=1}^{2} \frac{L_{11}^i \sin^2\theta_i + L_{22}^i \cos^2\theta_i}{L_{11}^i L_{22}^i} \\ D_{22} &= \sum_{i=1}^{2} \frac{L_{11}^i \cos^2\theta_i + L_{22}^i \sin^2\theta}{L_{11}^i L_{22}^i} \\ D_{12} &= \sum_{i=1}^{2} \frac{\sin\theta_i \cos\theta_i (L_{11}^i - L_{22}^i)}{L_{11}^i L_{22}^i} \\ W_{21} &= \frac{S_{12}^2}{L_{22}^2} - \frac{S_{12}^1}{L_{22}^1} \end{aligned} \tag{9}$$

In eqn (9), θ_1 and θ_2 denote the rotation angles; and superscripts 1 and 2 on L_{11}, L_{22}, and S_{12} denote the upper and lower materials, respectively.

The oscillation index ε_α and eigenvector matrix $\mathbf{\Lambda}$ are given explicitly by

$$\varepsilon_1 = \varepsilon \frac{1}{2\pi} \ln \frac{1+\beta}{1-\beta}, \quad \varepsilon_2 = -\varepsilon, \quad \varepsilon_3 = 0, \quad \beta = \frac{W_{21}\,\mathrm{sgn}(W_{21})}{A} \tag{10}$$

$$\mathbf{\Lambda} = \begin{bmatrix} (-iA\,\mathrm{sgn}(W_{21}) - D_{12})/\sqrt{2D_{11}A} & (iA\,\mathrm{sgn}(W_{21}) - D_{12})/\sqrt{2D_{11}A} & 0 \\ \sqrt{D_{11}}/\sqrt{2A} & \sqrt{D_{11}}/\sqrt{2A} & 0 \\ 0 & 0 & 1/\sqrt{D_{33}} \end{bmatrix} \tag{11}$$

where

$$A = \sqrt{D_{11}D_{22} - D_{12}^2}$$

The sgn ( ) stands for sign function. From (9), it is noted that matrix **W** is the invariant of rotation θ. Hence, for bimaterial media consisting of identical orthotropic solids with different orientations on either side, no oscillation is present from (10), and the crack-tip stress field has the standard square root singularity. This class of bimaterial interfacial cracks has been studied in great detail by Bassani and Qu (1989). In the present study, we consider bimaterial media consisting of two dissimilar orthotropic solids with arbitrary rotations in the x_1–x_2 plane.

With the explicit expression of $\mathbf{\Lambda}$ and ε, the near-tip stresses and crack surface relative displacements in (3) and (4) can be stated explicitly in terms of stress intensity factors. The near-tip stresses ahead of the crack tip are

$$\sigma_{22} = \frac{1}{\sqrt{2\pi r}}\left[\left(Re(\xi) - \frac{D_{12}}{A} Im(\xi)\,\mathrm{sgn}(W_{21})\right)K_{\mathrm{I}} - \frac{D_{11}}{A} Im(\xi)\,\mathrm{sgn}(W_{21})K_{\mathrm{II}}\right] \tag{12}$$

$$\sigma_{12} = \frac{1}{\sqrt{2\pi r}}\left[\frac{D_{22}}{A} Im(\xi)\,\mathrm{sgn}(W_{21})K_{\mathrm{I}} + \left(Re(\xi) + \frac{D_{12}}{A}\right)K_{\mathrm{II}}\right] \tag{13}$$

$$\sigma_{23} = \frac{1}{\sqrt{2\pi r}} K_{\mathrm{III}} \tag{14}$$

The near-tip relative crack displacements are

$$\Delta u_1 = \sqrt{\frac{2r}{\pi}}\left[(A\, Im(\eta)\,\mathrm{sgn}(W_{21}) + D_{12}\, Re(\eta))K_{\mathrm{I}} + D_{11}\, Re(\eta)K_{\mathrm{II}}\right] \tag{15}$$

$$\Delta u_2 = \sqrt{\frac{2r}{\pi}}\left[D_{22}\, Re(\eta)K_{\mathrm{I}} + (-A\, Im(\eta)\,\mathrm{sgn}(W_{21}) + D_{12}\, Re(\eta))K_{\mathrm{II}}\right] \tag{16}$$

$$\Delta u_3 = \frac{2r}{\pi}\sqrt{D_{33}} K_{\mathrm{III}} \tag{17}$$

In (12)–(17), complex constants ξ and η are defined as

$$\eta = \frac{\xi}{(1+2i\varepsilon)\cosh \pi\varepsilon}, \quad \xi = (r/2a)^{i\varepsilon}$$

It is seen from the above near-tip solutions that the in-plane and out-of-plane deformations are uncoupled, and only in-plane deformation possesses oscillatory behavior. Since the oscillatory field is the major interest in our study, only in-plane problems are studied here.

3. FINITE EXTENSION STRAIN ENERGY RELEASE RATES

Due to the oscillatory nature, the individual Mode I and Mode II strain energy release rates do not exist for interfacial cracks in isotropic media and anisotropic media. Instead, the strain energy release rates for a finite crack extension Δa were introduced by Sun and Qian (1997) for interfacial cracks in isotropic media and by Qian and Sun (1995, 1997) for interfacial cracks in composite laminates to calculate stress intensity factors. We extend this approach to the interfacial cracks between two dissimilar orthotropic media having a plane of material symmetry coinciding parallel to the x_1–x_2 plane.

If we allow a finite crack extension Δa (and thus a finite crack closure length) in Irwin's crack closure integrals, i.e.,

$$\hat{G}_{\mathrm{I}} = \frac{1}{2\Delta a}\int_0^{\Delta a} \sigma_{22}(r,0)\Delta u_2(\Delta a - r, \pi)\,\mathrm{d}r \tag{18}$$

$$\hat{G}_{\mathrm{II}} = \frac{1}{2\Delta a}\int_0^{\Delta a} \sigma_{12}(r,0)\Delta u_1(\Delta a - r, \pi)\,\mathrm{d}r \tag{19}$$

these integrals can be evaluated without ambiguity. Substituting the near-tip stresses and displacements from equations (12), (13), (15), and (16) and (18) and (19), and using the following integral identities

$$\int_0^{\Delta a} \left(\frac{x}{\Delta a - x}\right)^{(1/2)-i\varepsilon} dx = \frac{\pi \Delta a}{\cosh \pi\varepsilon}\left(\frac{1}{2} - i\varepsilon\right)$$

$$\int_0^{\Delta a} \sqrt{\frac{\Delta a - x}{x}} \left(\frac{\Delta a - x}{2a}\right)^{-i\varepsilon} \left(\frac{x}{2a}\right)^{-i\varepsilon} dx = \frac{\sqrt{\pi}}{2}\Delta a \left(\frac{\Delta a}{4a}\right)^{-2i\varepsilon} \frac{\Gamma(1/2 - i\varepsilon)}{\Gamma(1 - i\varepsilon)}$$

we obtain $\hat{G}_{\mathrm{I}}$ and $\hat{G}_{\mathrm{II}}$ in terms of stress intensity factors as

$$\hat{G}_{\mathrm{I}} = a^+ K_{\mathrm{I}}^2 + b^+ K_{\mathrm{II}}^2 + c^+ K_{\mathrm{I}} K_{\mathrm{II}} \tag{20}$$

$$\hat{G}_{\mathrm{II}} = a^- K_{\mathrm{I}}^2 + b^- K_{\mathrm{II}}^2 + c^- K_{\mathrm{I}} K_{\mathrm{II}} \tag{21}$$

$$G = \hat{G}_{\mathrm{I}} + \hat{G}_{\mathrm{II}} = \frac{1}{4\cosh^2 \pi\varepsilon}[D_{22} K_{\mathrm{I}}^2 + D_{11} K_{\mathrm{II}}^2 + 2D_{12} K_{\mathrm{I}}^2 K_{\mathrm{II}}^2] \tag{22}$$

where

$$a^{\pm} = \frac{1}{8A\cosh^2 \pi\varepsilon}\left[\frac{\pm(v+\bar{v})(A^2 - D_{12}^2) + 2AD_{11}D_{22} \mp 2AD_{12}i(v-\bar{v})\,\mathrm{sgn}(W_{21})}{2D_{11}}\right]$$

$$b^{\pm} = \frac{1}{8A\cosh^2 \pi\varepsilon}\left[\frac{D_{11}(2A \mp (v+\bar{v}))}{2}\right]$$

$$c^{\pm} = \frac{1}{8A\cosh^2 \pi\varepsilon}[D_{12}(2A \mp (v+\bar{v}) \mp Ai(v-\bar{v})\,\mathrm{sgn}(W_{21})]$$

The complex constant v appears in the above coefficients as

$$v = d\left(\frac{\Delta a}{4a}\right) \tag{23}$$

where

$$d = \frac{(A + D_{12}\,\mathrm{sgn}(W_{21})i)\cosh \pi\varepsilon}{\sqrt{\pi}(1 - 2i\varepsilon)} \frac{\Gamma(1/2 - i\varepsilon)}{\Gamma(1 - i\varepsilon)}$$

in which, Γ is the gamma function. It is clearly seen that the finite extension strain energy release rates $\hat{G}_{\mathrm{I}}$ and $\hat{G}_{\mathrm{II}}$ do not converge when Δa approaches zero due to the presence of the oscillation term $(\Delta a/4a)^{-2i\varepsilon}$. This oscillation term is also present in the strain energy release rates for interfacial cracks in isotropic media (Sun and Jih, 1987). Nevertheless, the total strain energy release rate is still well defined.

4. MODE MIXITY

Due to the oscillatory singularity in the near tip stress field, K_{I} and K_{II} for interfacial racks cannot be uniquely associated with Mode I and Mode II fracture as defined in homogeneous media. Nevertheless, K_{I} and K_{II} still represent two different modes of fracture action, and their respective amounts of participation in fracture can be reflected by the mode mixity angle defined by

$$\psi_K = \tan^{-1}\left(\frac{K_{II}}{K_I}\right) \tag{24}$$

If we adopt a definition of stress intensity factor that includes the phase effect of crack length such as Hutchinson's (1987), the mode mixity ψ_K would be ambiguous due to the nature of oscillatory singularity of the interfacial crack. In order to define the mixed mode fracture toughness unambiguously, Rice (1988) suggested a definition of stress intensity factor of the classical type by introducing the use of a specific distance $\hat{r}$ from the crack tip to define K. The individual stress intensity factors K_I and K_{II} based on location $\hat{r} = 2a$ can be used to define the mixed mode fracture condition as

$$G(\psi_K) = G_c(\psi_K) \tag{25}$$

As shown in eqn (2), ψ_K obtained at one $\hat{r}$ can be converted to those at a different $\hat{r}$. Hence, there are no restrictions on the selection of $\hat{r}$. The above fracture criterion was suggested by Hutchinson (1990) for the case $\varepsilon = 0$ of interfacial cracks in isotropic media. When $\varepsilon = 0$, G_I and G_{II} are well defined and the mode mixity ψ can also be fully expressed in terms of the strain energy release rates as

$$\psi_G = \tan^{-1}\left(\frac{G_{II}}{G_I}\right) \tag{26}$$

However, the mode mixity cannot be expressed for interfacial cracks in terms of the ratio of $\hat{G}_{II}$ and $\hat{G}_I$ due to their Δa-dependency and their non-convergent nature (as $\Delta a \to 0$). In view of the foregoing, we replaced K_I and K_{II} in (20) and (21) by total strain energy release rate G through manipulation of (20)–(22). Finally, we can rewrite the finite extension strain energy release rates $\hat{G}_I$ and $\hat{G}_{II}$ in terms of G and mode mixity ψ_K as

$$\hat{G}_I = \frac{1}{2}G + \frac{|d|G}{2A}\left[\cos(\alpha-\phi)\cos\left(2\varepsilon\ln\frac{\Delta a}{4a}\right) + \sin(\alpha-\phi)\sin\left(2\varepsilon\ln\frac{\Delta a}{4a}\right)\right] \tag{27}$$

$$\hat{G}_{II} = \frac{1}{2}G - \frac{|d|G}{2A}\left[\cos(\alpha-\phi)\cos\left(2\varepsilon\ln\frac{\Delta a}{4a}\right) + \sin(\alpha-\phi)\sin\left(2\varepsilon\ln\frac{\Delta a}{4a}\right)\right] \tag{28}$$

where $|d|$ and α are related to the complex constant d in (23) as

$$d = |d|e^{i\alpha}$$

and

$$\phi = \tan^{-1}\frac{2A\,\mathrm{sgn}(W_{21})(D_{12} + D_{11}\tan\psi_K)}{A^2 - D_{12}^2 - D_{11}^2\tan^2\psi_K - 2D_{11}D_{12}\tan\psi_K} \tag{29}$$

We introduce Δa-independent quantities $\bar{G}_I$ and $\bar{G}_{II}$ by extracting the amplitudes of the oscillation terms in eqns (27) and (28), respectively. Thus,

$$\begin{aligned}\bar{G}_I &= \frac{1}{2}G + \frac{|d|G}{2A}[\cos(\alpha-\phi) + \sin(\alpha-\phi)] \\ &= \frac{1}{2}G + \frac{\sqrt{2}|d|G}{2A}\cos\left(\alpha-\phi-\frac{\pi}{4}\right)\end{aligned} \tag{30}$$

$$\bar{G}_{II} = \frac{1}{2}G - \frac{|d|G}{2A}[\cos(\alpha-\phi)+\sin(\alpha-\phi)]$$
$$= \frac{1}{2}G - \frac{\sqrt{2}|d|G}{2A}\cos\left(\alpha-\phi-\frac{\pi}{4}\right) \tag{31}$$

Solving eqns (27) and (28) for $(\alpha-\phi)$ and substituting it into (30) and (31), we obtain $\bar{G}_I$ and $\bar{G}_{II}$ in terms of $\hat{G}_I$ and $\hat{G}_{II}$ as

$$\bar{G}_I = \frac{1}{2}G + \frac{|d|G}{\sqrt{2}A}\cos\left(2\varepsilon\ln\frac{\Delta a}{4a} - \frac{\pi}{4} + \cos^{-1}\left(A\frac{\hat{G}_I - \hat{G}_{II}}{|d|G}\right)\right) \tag{32}$$

$$\bar{G}_{II} = \frac{1}{2}G - \frac{|d|G}{\sqrt{2}A}\cos\left(2\varepsilon\ln\frac{\Delta a}{4a} - \frac{\pi}{4} + \cos^{-1}\left(A\frac{\hat{G}_I - \hat{G}_{II}}{|d|G}\right)\right) \tag{33}$$

Although the expressions of $\bar{G}_I$ and $\bar{G}_{II}$ in (32) and (33) contain the oscillation term $\ln(\Delta a/4a)$, it should be cancelled out by the term involving $\hat{G}_I$ and $\hat{G}_{II}$. Thus, $\bar{G}_I$ and $\bar{G}_{II}$ are non-oscillatory and well defined as $\Delta a \to 0$. Using the relations derived in eqns (32) and (33), $\bar{G}_I$ and $\bar{G}_{II}$ are obtained through the calculation of $\hat{G}_I$ and $\hat{G}_{II}$ by the crack closure method. Meanwhile, the mode mixity can also be represented unambiguously by

$$\psi_{\bar{G}} = \tan^{-1}\left(\frac{\bar{G}_{II}}{\bar{G}_I}\right) \tag{34}$$

Following the definition of $\psi_{\bar{G}}$ in (34) and using eqns (30)–(31), the mode mixity quantities ψ_K and $\psi_{\bar{G}}$ are related by

$$\tan\psi_{\bar{G}} = \frac{\sqrt{2}A - 2|d|\cos(\alpha-\phi-\pi/4)}{\sqrt{2}A + 2|d|\cos(\alpha-\phi-\pi/4)} \tag{35}$$

Note that, in (35), ϕ is a function of ψ_K. Hence, the fracture criterion (25) can be expressed alternatively in terms of G and $\psi_{\bar{G}}$, or $\bar{G}_I$ and $\bar{G}_{II}$.

Beuth (1995) also proposed non-oscillatory strain energy release rates for the purpose of mode separation for interfacial cracks in orthotropic media. In his formulation, the individual strain energy release rates G_I and G_{II} were expressed as

$$G_I = \lim_{\Delta a\to 0}\frac{1}{4\Delta a}[\Phi_1 + \Phi_2] \tag{36}$$

$$G_{II} = \lim_{\Delta a\to 0}\frac{1}{4\Delta a}[\Phi_1 - \Phi_2] \tag{37}$$

where

$$\Phi_1 = \frac{H_{11}}{2\cosh^2\pi\varepsilon}K\bar{K}\Delta a$$
$$\Phi_2 = \frac{H_{11}}{2\sqrt{\pi}\cosh\pi\varepsilon}Re\left[\frac{(Kh^{i\varepsilon})^2}{1+2i\varepsilon}\left(\frac{\Delta a}{2h}\right)^{2i\varepsilon}\frac{\Gamma(\frac{1}{2}+i\varepsilon)}{\Gamma(1+i\varepsilon)}\right]$$

The stress intensity factor K follows Hutchinson's definition (1987) which involves the logarithm of crack length, h is a normalized constant for the crack extension length Δa, and H_{11} is a material constant. It is clearly seen that the Φ_1 does not involve oscillation, while Φ_2 has the oscillation term $(\Delta a/2h)^{2i\varepsilon}$. In order to isolate the oscillatory behavior of the energy release quantities, Beuth (1995) introduced Φ_2' by excluding the oscillation term $(\Delta a/2h)^{2i\varepsilon}$ in Φ_2 as

$$\Phi_2' = \frac{H_{11}}{2\sqrt{\pi}\cosh\pi\varepsilon} Re\left[\frac{(Kh^{i\varepsilon})^2}{1+2i\varepsilon}\frac{\Gamma(\frac{1}{2}+i\varepsilon)}{\Gamma(1+i\varepsilon)}\right]$$

The non-oscillatory strain energy release rates G_1 and G_2 introduced by Beuth are expressed as

$$G_1 = \lim_{\Delta a\to 0}\frac{1}{4\Delta a}[\Phi_1+\Phi_2'] \tag{38}$$

$$G_2 = \lim_{\Delta a\to 0}\frac{1}{4\Delta a}[\Phi_1-\Phi_2'] \tag{39}$$

The major difference between the non-oscillatory strain energy release rates $\bar{G}_{\mathrm{I}}$ and $\bar{G}_{\mathrm{II}}$, and G_1 and G_2 is that the latter simply exclude the oscillation term while the former extracts the amplitude of the oscillation term.

It was suggested by Sun and Qian (1997) that the near-tip oscillation zone r_0 be expressed in terms of mode mixity such as ψ_K for interfacial cracks in isotropic media. The size of the oscillation zone can be used as an estimate of the actual contact zone and as an evaluation of the validity of loading by small scale contact zone concept. Letting near-tip crack loading displacements in (16) vanish, we have

$$r_0 = 2a\exp\left[-\frac{1}{\varepsilon}\left(\tilde{\psi}-\frac{\pi}{2}\right)\right] \tag{40}$$

and

$$\tilde{\psi} = \tan^{-1}\frac{(A\,\mathrm{sgn}(W_{21})-2\varepsilon D_{12})\tan^{-1}\psi_K-2\varepsilon D_{22}}{(2A\varepsilon\,\mathrm{sgn}(W_{21})+D_{12})\tan^{-1}\psi_K+D_{22}} \tag{41}$$

The value of $r_0/2a$ can be used to check whether the small scale contact concept can be applied. To insure that the crack tip state is K-dominated, Rice (1988) suggested that $r_0/2a \leqslant 0.01$, or equivalently,

$$\exp\left[-\frac{1}{\varepsilon}\left(\tilde{\psi}+\frac{\pi}{2}\right)\right] \leqslant 0.01 \tag{42}$$

Therefore, the limitation of loading can be evaluated through mode mixity angle ψ_K.

5. CALCULATION OF STRESS INTENSITY FACTORS USING FINITE ELEMENT ANALYSIS

For interfacial cracks in isotropic media, analytical methods for solving the stress intensity factors for interfacial crack problems are limited to a few special cases due to the inherent mathematical difficulties. Numerical methods such as the finite element method are needed to calculate the stress intensity factors for interfacial cracks in bodies of finite dimensions under general loading conditions. For interfacial cracks in isotropic media, Lin and Mar (1976) employed a special singular bi-material crack element to model the crack tip region. This crack element was derived based on the interfacial crack tip stress field and the hybrid formulation. Matos *et al.* (1989) proposed a method for calculating stress intensity factors based on the evaluation of the J-integral. Individual stress intensity factors are obtained from further calculation of J perturbed by small increments of stress intensity factors.

Sun and Qian (1997) have recently proposed two new methods to compute stress intensity factors; one is called the energy method which calculates the finite extension strain energy release rates $\hat{G}_{\mathrm{I}}$ and $\hat{G}_{\mathrm{II}}$ by the modified crack closure method (Rybicki and Kanninen, 1977) using finite element analysis. With the established relations between $\hat{G}_i$ and K_i ($i = \mathrm{I}, \mathrm{II}$), the stress intensity factors are obtained. The other is the displacement ratio method which evaluates the ratio of stress intensity factors based on the crack surface displacement ratio. The additional equation needed to determine the stress intensity factors is provided by the total strain energy release rate.

The quadratic relations between $\hat{G}_i$ and K_i in eqns (20) and (21) indicate that for one set of $\hat{G}_i$ we may have two sets of solution for K_i. However, there is only one set of roots for K_i which corresponds to the displacement field. We may utilize the relative crack surface displacements obtained from the finite element analysis to select the correct K_i. Specifically, the correct values of K_{I} and K_{II} obtained from $\hat{G}_{\mathrm{I}}$ and $\hat{G}_{\mathrm{II}}$, after substitution into eqns (15) and (16), should yield relative crack tip opening displacements that agree with the finite element result.

It may appear that the linear relations in eqns (15) and (16) may be used to calculate K_{I} and K_{II} based on the relative crack surface displacements. However, it is found that the stress intensity factors extracted in this manner based on the finite element result usually are not accurate.

Using eqns (15) and (16), the ratio of $K_{\mathrm{II}}/K_{\mathrm{I}}$ can be expressed in terms of crack surface displacement ratio $\Delta u_2/\Delta u_1$, and the stress intensity factors K_{I} and K_{II} can be obtained using this ratio ($K_{\mathrm{II}}/K_{\mathrm{I}}$) together with eqn (22) where the total strain energy release rate can be obtained accurately from the crack closure method. It is easy to see that the displacement ratio method is more convenient to perform than the energy method.

6. NUMERICAL EXAMPLES

The analytical solution of stress intensity factors for an infinite bimaterial medium of orthotropic solids containing an interfacial crack of length $2a$ subjected to uniform remote tensile σ_{22}^{∞} and shear loading σ_{12}^{∞} was originally derived by Hwu (1993b) in matrix form. From Hwu's solution, the stress intensity factors can be expressed explicitly as

$$K_{\mathrm{I}} = \sqrt{\pi a}\left[\left(1+\frac{2\varepsilon D_{12}}{A}\operatorname{sgn}(W_{21})\right)\sigma_{22}^{\infty}-\frac{2\varepsilon D_{11}}{A}\operatorname{sng}(W_{21})\sigma_{12}^{\infty}\right] \tag{43}$$

$$K_{\mathrm{II}} = \sqrt{\pi a}\left[\frac{2\varepsilon D_{22}}{A}\operatorname{sgn}(W_{21})\sigma_{22}^{\infty}+\left(1+\frac{2\varepsilon D_{12}}{A}\operatorname{sgn}(W_{21})\right)\sigma_{12}^{\infty}\right] \tag{44}$$

Appropriate uniform stresses $\sigma_{11}^{(1)}$ and $\sigma_{11}^{(2)}$ (see Fig. 2) are required to satisfy for displacement continuity along the interface, i.e.,

$$\varepsilon_{11}^{(1)} = \varepsilon_{11}^{(2)} \tag{45}$$

in which superscripts 1 and 2 denote the upper and lower materials, respectively. This condition can be expressed in terms of stress components and the applied stresses σ_{22}^{∞} and σ_{12}^{∞} as

$$\hat{S}_{1111}^{(1)}\sigma_{11}^{(1)}-\hat{S}_{1111}^{(2)}\sigma_{11}^{(2)} = (\hat{S}_{1122}^{(2)}-\hat{S}_{1122}^{(1)})\sigma_{22}^{\infty}+(\hat{S}_{1121}^{(2)}-\hat{S}_{1112}^{(1)})\sigma_{12}^{\infty} \tag{46}$$

where $\hat{S}_{ijkl}$ are reduced material compliancies which are related to material compliancies S_{ijkl} as

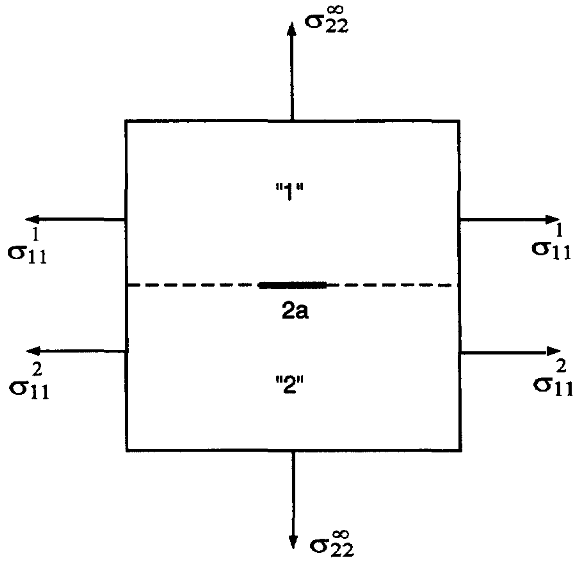

Fig. 2. Infinite bimaterial medium containing an interfacial crack of length of $2a$ subject to remote tensile loading.

$$\hat{S}_{ijkl} = S_{ijkl} - S_{ij33}S_{33kl}/S_{3333} \quad (i,j,k,l = 1,2,3)$$

The problem studied by Hwu (1993b) is taken to verify the present finite element analysis procedure. Only uniform remote tensile loading as shown in Fig. 2 is considered. The size of the bimaterial medium is taken to be 100 m × 100 m with a 2 m interfacial crack. The tensile loading σ_{22}^{∞} is assumed to be 1 N/m^2. Two orthorhombic crystalline materials are selected: Argonite for the upper material and Topaz for the lower material. Various material orientations $[\theta_1/\theta_2]$ are considered. The elastic constants of these two materials are listed in Table 1.

The present finite element calculations were carried out with the ABAQUS code. The eight-node isoparametric element was used for the analysis. The modified crack-closure method, based on the nodal forces and displacements, was used to compute the finite extension strain energy release rates $\hat{G}_{\mathrm{I}}$ and $\hat{G}_{\mathrm{II}}$.

The accuracy of stress intensity factors calculated with the energy method depends on the accuracy of strain energy release rate calculation using the modified crack closure method. Sun and Jih (1987) performed a convergence test for $\hat{G}_{\mathrm{I}}$ and $\hat{G}_{\mathrm{II}}$ with various crack extensions $\Delta a/a$ for an interfacial crack between two isotropic solids. They showed that in the range of $\Delta a/a$ from $2.5 \cdot 10^{-3}$ to $5 \cdot 10^{-2}$, the calculated strain energy release rates by the modified crack closure method agreed well with the analytical solutions. However, it is obvious that the choice of $\Delta a/a$ cannot be totally arbitrary. If very small values of $\Delta a/a$ are used, the oscillatory behavior of the near-tip field may lead to unreliable stresses and displacement in the finite element analysis. Moreover, small values of $\Delta a/a$ require extra meshing effort and a large number of elements. On the other hand, large values of $\Delta a/a$ will invalidate the neartip relations based on which the relations between the finite extension strain energy release rates and the stress intensity factors are derived.

Table 1. Elastic constants (10^{11} N/m^2) for two orthorhombic crystalline materials

No.	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}
1	1.5958	0.3663	0.0197	0.8697	0.1591	0.8503	0.4132	0.2564	0.4274
2	2.8136	1.2582	0.8464	3.4895	0.8815	2.9452	1.0811	1.3298	1.3089

Material No. 1: Argonite; Material No. 2: Topaz.

In view of the foregoing, the ratio $\Delta a/a$ was chosen to be 0.01 in calculating $\hat{G}_{\mathrm{I}}$ and $\hat{G}_{\mathrm{II}}$. The calculated finite extension strain energy release rates and calculated stress intensity factors from the energy method as well as those from the analytical solutions (43) and (44) are listed in Tables 2 and 3. From Table 2, it is found that the discrepancies between the two solutions are well below 1% except for the [−45/45] case, for which the discrepancies are slightly greater. Comparison of the stress intensity factors is given in Table 3. It is noted that the errors in stress intensity factors for the [−45/45] case are below 1% despite slightly larger errors in strain energy release rates.

The calculation of stress intensity factors based on the displacement ratio method is also performed for the above problem. Note that, along the crack plane and near the crack tip, the finite element mesh was kept uniform with the element size 1% of the half crack

Table 2. Strain energy release rates for a 100 m × 100 m bimaterial plane strain medium subject to remote tensile loading $\sigma_{22}^{\infty} = 1\ \mathrm{N/m^2}$ by the energy method

	$\hat{G}_{\mathrm{I}}$			$\hat{G}_{\mathrm{II}}$			G
$[\theta_1/\theta_2]$	Exact eqn (27)	F.E.M. present	Error (%)	Exact eqn (28)	F.E.M. present	Error (%)	Error (%)
[0/0]	1.9727E-11	1.9701E-11	0.13	1.8772E-12	1.8900E-12	0.68	0.06
[0/90]	2.0229E-11	2.0201E-11	0.14	1.9385E-12	1.9519E-12	0.69	0.07
[90/0]	1.5744E-11	1.5717E-11	0.17	1.5081E-12	1.5131E-12	0.33	0.12
[0/−45]	2.0188E-11	2.0085E-11	0.51	1.8318E-12	1.8466E-12	0.81	0.40
[−45/45]	1.6413E-11	1.6619E-11	1.61	2.2674E-12	2.3029E-12	2.01	1.62

Table 3. Stress intensity factors for a 100 m × 100 m bimaterial plane strain medium subject to remote tensile loading σ_{22}^{∞} by the energy method

	$K_{\mathrm{I}}/\sqrt{\pi a}\sigma_{22}^{\infty}$			$K_{\mathrm{II}}/\sqrt{\pi a}\sigma_{22}^{\infty}$		
$[\theta_1/\theta_2]$	Exact eqn (43)	F.E.M. energy method	Error (%)	Exact eqn (44)	F.E.M. energy method	Error (%)
[0/0]	1	0.9998	0.02	0.1169	0.1157	1.00
[0/90]	1	0.9997	0.03	0.1207	0.1194	1.07
[90/0]	1	0.9995	0.05	0.094	0.093	1.06
[0/−45]	1.0015	0.9998	0.17	0.1188	0.1166	1.85
[−45/45]	0.9866	0.9945	0.80	0.1074	0.1076	0.19

Table 4. Relative errors in stress intensity factors by the displacement ratio method

$[\theta_1/\theta_2]$	Element no. (%)	First	Second	Third	Fourth	Fifth
[0/0]	Error (K_{I})	0.06	0.03	0.04	0.04	0.04
	Error (K_{II})	8.19	0.31	1.56	1.37	1.58
[0/90]	Error (K_{I})	0.09	0.00	0.02	0.02	0.02
	Error (K_{II})	7.82	0.41	1.61	1.42	1.62
[90/0]	Error (K_{I})	0.04	0.08	0.08	0.08	0.08
	Error (K_{II})	9.97	1.83	1.53	1.15	1.40
[0/−45]	Error (K_{I})	0.13	0.22	0.23	0.23	0.23
	Error (K_{II})	5.79	1.35	2.28	1.94	2.05
[−45/45]	Error (K_{I})	0.77	0.80	0.80	0.80	0.80
	Error (K_{II})	13.5	0.33	0.20	0.40	0.22

size a. Crack surface displacements at various locations are taken to compute the displacements ratios. The results are presented in Table 4. It is evident that the stress intensity factors obtained with this method are in excellent agreement with the analytical solutions if the nodal displacements are taken at least two elements away from the crack tip.

7. CONCLUSION

Two finite element based methods have been shown to be accurate for calculating stress intensity factors for interfacial cracks between two orthotropic solids. The first method involves the evaluation of strain energy release rates $\hat{G}_i$ associated with a finite crack extension using the modified crack closure method. From the explicit near-tip stress and displacement fields, the relations between the finite extension strain energy release rates and stress intensity factors are derived from which stress intensity factors are determined. The second method utilizes the relation between the crack surface displacement ratio and the ratio of stress intensity factors to determine the stress intensity factors. It was shown through numerical examples that both methods are quite efficient and accurate. The displacement ratio method, perhaps, is more convenient to use in conjunction with the finite element analysis. The non-oscillatory quantities $\bar{G}_{\mathrm{I}}$ and $\bar{G}_{\mathrm{II}}$ derived from $\hat{G}_{\mathrm{I}}$ and $\hat{G}_{\mathrm{II}}$ can be used to give an alternative expression for the interfacial crack mode mixity. Thus, the fracture criterion can also be given in terms of the total strain energy release rate G and the ratio $\bar{G}_{\mathrm{II}}/\bar{G}_{\mathrm{I}}$ without calculating the stress intensity factors.

Acknowledgements—This work was supported in part by a Purdue Research Foundation grant and in part by the Office of Naval Research Grant no. N00014-90-J-1666 for which Dr Y. D. S. Rajapakse was the grant monitor.

REFERENCES

Barnett, D. M. and Lothe, J. (1973) Synthesis of the sextic and the integral formalism for dislocation, Green's function and surface waves in anisotropic elastic solids. *Phys. Norv.* **7**, 13–19.

Bassani, J. L. and Qu, J. (1989) Finite crack on bimaterial and bicrystal s. *Journal of the Mechanics and Physics of Solids* **37**, 435–453.

Beuth, J. L. (1995) Separation of crack extension modes in orthotropic delamination models. NASA Technical Memorandum 109,180.

Clements, D. L. (1971) A crack between dissimilar anisotropic media. *International Journal of Engineering Science* **9**, 257–265.

Comninou, M. (1977) The crack. *Journal of Applied Mechanics* **44**, 631–636.

Comninou, M. (1978) The crack in a shear field. *Journal of Applied Mechanics* **45**, 287–290.

Dongye, C. and Ting, T. C. T. (1989) Explicit expressions of Barnett–Lothe tensors and their associated tensors for orthotropic materials. *Quarterly of Applied Mathematics* **47**, 732–734.

England, A. H. (1965) A crack between dissimilar media. *Journal of Applied Mechanics* **32**, 400–402.

Gao, H., Abbudi, M. and Barnett, D. M. (1992) Interfacial crack-tip field in anisotropic elastic solids. *Journal of Applied Mechanics* **40**, 393–416.

Gotoh, M. (1967) Some problems of bonded anisotropic plates with crack along the bond. *International Journal of Fracture* **3**, 253–265.

Hutchinson, J. W. (1990) Mixed mode fracture mechanics of s. *Metal-Ceramic, Acta-Scripta Metallurgica* **4**, 295–306.

Hutchinson, J. W., Mear, M. and Rice, J. R. (1987) Crack paralleling in between dissimilar materials. *Journal of Applied Mechanics* **54**, 828–832.

Hwu, C. (1993a) Explicit solutions for co-linear crack problems. *International Journal of Solids and Structures* **3**, 301–312.

Hwu, C. (1993b) Fracture parameters for the orthotropic bimaterial cracks. *Eng. Fract. Mech.* **45**, 89–97.

Lin, K. Y. and Mar, J. W. (1976) Finite element analysis of stress intensity factors for cracks at a bi-material. *International Journal of Fracture* **12**, 521–531.

Malysev, B. M. and Salganik, R. L. (1965) The strength of adhesive joints using the theory of cracks. *International Journal of Fracture* **1**, 114–127.

Matos, P. P. L., McMeeking, R. M., Charalamibides, P. G. and Drory, D. (1989) A method for calculation of stress intensities in bimaterial fracture. *International Journal of Fracture* **40**, 235–254.

Qian, W. and Sun, C. T. (1995) Evaluation of stress intensity factor of a crack between dissimilar general orthotropic solids. *24th Midwestern Mechanics Conference*. 1–4 October, pp. 125–127. Ames, Iowa.

Qian, W. and Sun, C. T. (1997) Calculation of stress intensity factors for interlaminar cracks in composite laminates. *Comp. Sci. Technol.* **57**, 637–650.

Raju, I. S., Crews, J. H. Jr. and Aminpour, M. A. (1988) Convergence of strain energy release rate components for edge-delaminated composite laminates. *Eng. Fract. Mech.* **30**, 383–396.

Rice, J. R. (1988) Elastic fracture mechanics concepts for interfacial cracks. *Journal of Applied Mechanics* **55**, 98–103.

Rice, J. R. and Sih, G. C. (1965) Plane problems of cracks in dissimilar media. *Journal of Applied Mechanics* **32,** 418–423.

Rybicki, E. F. and Kanninen, M. F. (1977) A finite element calculation of stress intensity factors by a modified crack closure integral. *Eng. Fract. Mech.* **9,** 931–938.

Sun, C. T. and Jih, C. J. (1987) On strain energy release rate for interfacial cracks in bimaterial media. *Eng. Fract. Mech.* **28,** 13–20.

Sun, C. T.and Manoharan, M. G. (1989) Strain energy release rates of an interfacial crack between two orthotropic solids. *Journal of Composite Materials* **23,** 460–478.

Sun, C. T. and Qian, W. (1997) The use of finite extension strain energy release rates in fracture of interfacial cracks. *International Journal of Solids and Structures* **34,** 2595–2609.

Ting, T. C. T. (1986) Explicit solution and invariance of the singularities at a crack in anisotropic composites. *International Journal of Solids and Structures* **22,** 965–983.

Wang, S. S. and Choi, I. (1983a) The crack between dissimilar anisotropic materials. *Journal of Applied Mechanics* **50,** 169–178.

Wang, S. S. and Choi, I. (1983b) The crack between dissimilar anisotropic composites under mixed-mode loading. *Journal of Applied Mechanics* **50,** 179–183.

Williams, M. L. (1959) The stresses around a fault or crack in dissimilar media. *Bull. Seismol. Soc. Amer.* **49,** 199–204.

Willis, J. R. (1971) Fracture mechanics of interfacial cracks. *Journal of the Mechanics and Physics of Solids* **19,** 353–368.

Wu, K. C. (1990) Stress intensity factor and energy release rate for interfacial cracks between dissimilar anisotropic materials. *Journal of Applied Mechanics* **57,** 882–886.

APPENDIX

$\mathbf{\Lambda}$ and ε_α are related to the following eigenvalue problem given by Hwu (1993a):

$$(\mathbf{M}^* + e^{2i\pi\delta}\bar{\mathbf{M}}^*)\mathbf{\Lambda} = 0 \tag{A1}$$

where $\mathbf{M}^*$ is the bimaterial matrix defined as

$$\begin{aligned} \mathbf{M}^* &= \mathbf{D} - i\mathbf{W} \\ \mathbf{D} &= \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1} \\ \mathbf{W} &= \mathbf{S}_1\mathbf{L}_1 - \mathbf{S}_2\mathbf{L}_2 \end{aligned} \tag{A2}$$

where S_i and L_i are Barnett–Lothe's tensors (Barnett and Lothe 1973), which are composed of the elastic constants of material i ($i = 1, 2$). Here, the subscripts 1 and 2 denote upper and lower materials, respectively. The explicit solution for the eigenvalue δ is given by Ting (1986) as

$$\begin{aligned} &\delta_\alpha = -\frac{1}{2} + i\varepsilon_\alpha, \quad \alpha = 1, 2, 3, \\ &\varepsilon_1 = \varepsilon = \frac{1}{2\pi}\ln\frac{1+\beta}{1-\beta}, \quad \varepsilon_2 = -\varepsilon, \quad \varepsilon_3 = 0, \quad \beta = \left[-\frac{1}{2}\mathrm{tr}(\mathbf{W}\mathbf{D}^{-1})^2\right]^{1/2} \end{aligned} \tag{A3}$$

where tr stands for the trace of a matrix; ε is called the oscillation index. Matrix $\mathbf{\Lambda}$ is composed of three different eigenvectors, i.e..

$$\mathbf{\Lambda} = [\lambda_1 \quad \lambda_2 \quad \lambda_3] \tag{A4}$$

The eigenvectors λ_i are determined from the eigenvalue problem given in (A1) up to an arbitrary complex constant and were normalized by Hwu (1993a) as

$$\bar{\mathbf{\Lambda}}^T\mathbf{D}\mathbf{\Lambda} = I \tag{A5}$$